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Thinking Like a Mathematician

Michael K. Weiss and Deborah Moore-Russo

The moves that mathematicians use to generate new questions can also be used by teachers and students to tie content together and spur exploration.

What does it mean to think like a mathematician? One of the great paradoxes of mathematics education is that, although we mathematics teachers are immersed in mathematical work every day of our professional lives, most of us nevertheless have little experience with the kind of work that research mathematicians do. Our ideas of what doing mathematics looks like are based mainly on our own experiences as students. Creating opportunities for students to engage in the kind of flexible thinking that is characteristic of mathematicians' practice can be a challenge for all of us.

Research has shown that most teachers, long before we ever enroll in our first education courses, already have firmly established ideas about teaching in general and mathematics instruction in particular; moreover, the ideas that we bring with us into teacher education programs often form the foundation on which we eventually build our own teaching (Millsaps 2000; Skott 2001). Before we entered the classroom as professionals, we were students for years—what Lortie (1975) referred to as an "apprenticeship of observation." The mathematics courses we took in high school as well as in our teacher education programs form some of our last preprofessional encounters with mathematics. Unfortunately, experience suggests that, for many of us, knowing mathematics is synonymous with being adept at manipulating symbols (Thompson 1992) and following memorized, step-by-step procedures (Ball 1988). Hence, many teachers consider mathematics "a discipline with a priori rules and procedures that … students have to learn by rote" (Handal 2003, p. 54).

Because of our own experiences, we often do not expect our students "to develop mathematical meanings" or "to use meanings in their thinking" (Thompson 2008, p. 45). It is, sadly, still unusual for students and teachers to be engaged in mathematical activities that require—or even allow for—flexible mathematical thinking. How often are students asked to pose their own problems or generalize a solution for a specific problem to a broader class of problems? In short, what do our students really learn about what it means to do mathematics?

Perhaps the most important questions are these: How can we break this self-perpetuating cycle? How can we, as teachers, encourage flexible mathematical thought if we ourselves had little opportunity to think flexibly when we were students?

HOW DO MATHEMATICIANS DO WHAT THEY DO?

One way to prepare ourselves and others to foster flexible mathematical thinking is to focus on the practices and habits of mind of research mathematicians—those who strive to generate new and refine existing mathematical ideas and methods. What does it mean to act like a mathematician?

In seeking to answer this question, Weiss (2009) analyzed a collection of narratives written by and about research mathematicians. This analysis reveals the fundamentally generative nature of engaging in mathematics, in which problem posing (asking fruitful and difficult questions of oneself and others) plays a role just as important as problem solving. The key role of problem posing in mathematics instruction has long been recognized. Silver (1994) noted that problem posing is a prominent feature not only of mathematical activity; it also features heavily in inquiry-oriented instruction and can help create an environment in which students are more engaged.

For example, Brown and Walter's (2004) "what-if-not?" strategy is a relatively simple means of generating new problems and promoting mathematical curiosity. Rather than accept a problem as it reads, a teacher asks instead: "What if this problem were not restricted to its given conditions?"

Consider the following scenario from high school geometry: The teacher has demonstrated the construction of a perpendicular bisector of a segment using a compass and straightedge, and students have practiced the method. The teacher then uses "what-if-not?" by asking: "What if the left and right arcs were not drawn with equal compass openings? Would the result still be a perpendicular bisector?"

The "what-if-not?" strategy is not limited to exploring geometry. Suppose that in a unit on

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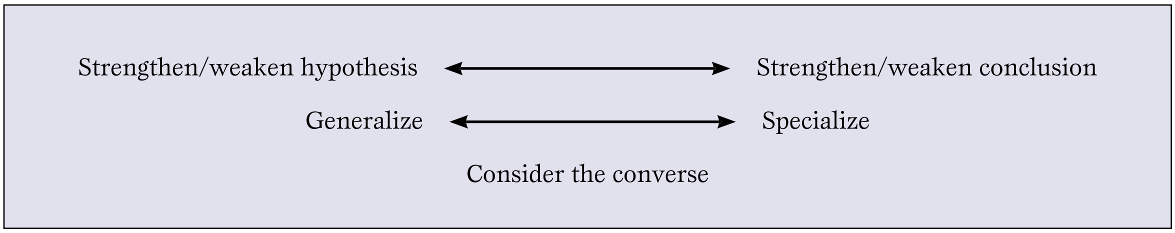

Fig. 1 Generating new problems from old ones can be accomplished using these strategies.

elementary statistics the teacher has provided the class with a set of three numbers (10, 20, and 30), and students have found that 20 is both the mean and the median of the set. Using "what-if-not?" the teacher could ask, "How would the mean and median change if the three numbers were not equally spaced?" More specifically, the teacher could ask, "How would the middle number have to be changed if we wanted the mean to be larger (or smaller) than the median?"

The "what-if-not?" strategy is one format for generating new questions. In a broader study of what research mathematicians care about and how they generate new problems from old ones, Weiss (2009) identified five "generative moves" for problem posing (see **fig. 1**). Here are some examples from geometry and algebra of how these moves can be used in the classroom and how common instructional activities can be changed slightly to include instances of authentic mathematical work.

Suppose that a class of geometry students has been studying the properties of triangles and has found that the three angle bisectors of any acute triangle always intersect in a single point. The following scenarios show how teachers can use the five generative moves to guide students in exploring the related mathematical terrain:

- The teacher might ask, "Does it really matter whether the triangle is acute or not?" Investigating this question could lead the class to conclude that the restriction to the acute case was unnecessary and that the conclusion holds for all triangles—a case of weakening the hypothesis, the first generative move shown in **figure 1**.
- The teacher might encourage students to strengthen the conclusion of what has been proven—for example, by offering, "Not only do they intersect at a single point, but that point is the center of a circle that can be inscribed in the triangle."
- The teacher might encourage students to generalize their findings—perhaps by asking, "What if we construct the angle bisectors of other polygons? Do they meet at a point? If not, what happens?"
- The teacher might ask students to specialize their findings—for example, by observing, "If you construct angle bisectors in an equilateral triangle, what else is true? For example, is the point of intersection equidistant from the vertices of the triangle?"
- If students observe this last property, they then might be encouraged to consider the converse question: "If the angle bisectors of a particular triangle meet at a point that is equidistant from the vertices of the triangle, does this result mean that the triangle in question must be equilateral?"

These examples illustrate how the five generative moves for problem posing can be used to describe and promote the practice of wondering mathematically about what is true, a core component of flexible mathematical thinking. More examples could be generated almost without limit by iterating and recombining these moves.

Again, the use of these generative moves is not limited to geometry. Suppose that a first-year algebra class is at the very beginning of its study of linear functions. The class has created a table of values listing eight pairs of numbers (x, y) that satisfy the equation $3x - 2y = 0$, perhaps through trial and error, by choosing values for one of the variables and then solving the resulting equation for the other, or through some combination of methods. Students have plotted these pairs of numbers on a graph and have noted that they all appear to lie on a straight line through the origin. What could happen next?

- Students might consider the converse question: Does every point on that line satisfy the same equation?
- Having come to believe that the answer to the question above is yes, students might then consider the converse again and ask: Does every point not on the line fail to satisfy the equation? (Although this statement is technically the inverse, not the converse, of the previous statement, the two are logically equivalent.)
- Once students have identified the points on the line as precisely the set of all pairs of values that satisfy the equation, students could attempt to strengthen the conclusion by saying more about the line—for example, that the line rises by 3 units for every horizontal change of 2 units.

- Students could continue by asking questions such as this one: How would the graph be changed if the right side of the equation were not zero but some other constant? Such questions are an application of the generalization move (and, it should be noted, a direct application of the "what-if-not?" template).

Although all algebra teachers address the same basic facts about linear functions, these examples are meant to emphasize how the generative moves can be used to tie these facts together into a cohesive set of inquiry-driven activities. Modeling and promoting flexible mathematical thinking for students does not necessarily involve learning new ideas. It can also play a role in helping us recast the ideas that we already teach, tying facts and procedures into a coherent narrative.

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STUDENTS AS RESEARCHERS

Many teachers include a focus on problem solving in their practice. Some of the ideas above were incorporated in the teaching of problem posing in a high school geometry class. As a year-end assessment, students were assigned a research project on the mathematical properties of *duals*—figures formed by joining the midpoints of adjacent sides of a polygon (see **fig. 2**). The instructions for the project stated, in part:

> This assignment is different from what you are probably accustomed to, because you have to *generate the questions*—not just answer them ... (*a*) to pose interesting questions; (*b*) to investigate your questions, by drawing lots of pictures, making measurements, and looking for patterns; (*c*) to formulate conjectures, and present evidence to support your conjectures; (*d*) when possible, to answer your questions, and *prove* (or *disprove*) your conjectures (at which point they stop being conjectures and become *theorems*).

Students were encouraged to report on their investigation of questions that they could not answer as well as those that they could.

Students produced a wide range of questions, conjectures, and results. Most proved that the dual of any triangle is a half-scale, 180°-rotation of the original triangle. Most also investigated the duals of special quadrilaterals. Many students did more than simply list a set of disconnected results; they used the generative moves to link their findings into a coherent theory.

One student began with an observation (and a proof) that the dual of a square is always a square. He then asked whether the converse is true: If the dual of a quadrilateral is square, must the "outside" quadrilateral be square as well? On finding a counterexample, he returned to his original finding and then weakened the hypothesis: What if the "outside" quadrilateral is not a square but just a rectangle or a rhombus? These questions led to the discovery that the dual of a rectangle is a rhombus (and vice versa) and, hence, that the *second dual* (the dual of the dual) of a rectangle or a rhombus is another figure of the same type. The student then strengthened the conclusion, proving that the second dual of a rectangle or a rhombus is actually similar to the original figure, with a scale factor of one-half. Finally, he weakened the hypothesis of this result, finding that any parallelogram is similar to its second dual.

Another student also found that the dual of a rectangle is a rhombus (and vice versa) but then went off in an entirely different direction. She wrote that while doodling in her notebook during her world history class (a fine example of mathematical wondering, although perhaps not one that her world history teacher would have approved of), she became curious about what it would mean to form the dual of a cube or another solid. Two ideas occurred to her: She could join the midpoint of each edge to the midpoints of the neighboring edges, or she could join the center of each face to the centers of the adjacent faces. Sketching both possibilities, she discovered that, using the second approach, the dual of a cube is an octahedron and the dual of an octahedron is

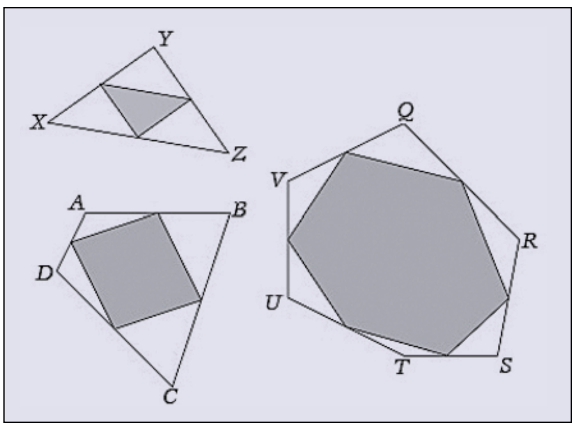

Fig. 2 Connecting consecutive midpoints of the sides of a polygon creates the figure's dual.

another cube—a result analogous to what she had already shown about rectangles and rhombi. She also found that the dual of a tetrahedron is an "upside-down" tetrahedron, analogous to what she already knew about triangles. On the basis of these findings, she decided that the second method is the "right" way to construct duals of solids—an impressive example of generalization as a generative move.

Not only did students demonstrate outstanding mathematical work; many of them took evident pride of ownership in what they had created. One wrote on the cover of her report, "The most FUN final ever!!!" Two others (separately) thanked the teacher when they turned in their work. In the years since teaching that class, the teacher has been surprised, on three separate occasions, by former students (now in college) reminiscing enthusiastically about the research projects. One student referred to it as "the first time I ever really got to *think* about math."

CONCLUSION

For our classrooms to become venues where students experience mathematical activities that foster flexible thinking, we need to change how we as teachers think about and do mathematics. Teachers at all levels should make the mechanisms of problem posing explicit and draw attention to how they can be used to navigate through open-ended problems. Through engagement in such mathematical activities, teachers and their students might come to view mathematics differently. By extension, these teachers might come to teach mathematics in a manner that is more authentic to actual mathematical practice.

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